

Hyperbolic Motion in the Finite-size Model of the Radiating Electron

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Within the framework of the finite-size model of the electron recently developed the case of hyperbolic motion is studied rigorously by means of an exact solution. The energy-momentum balance is discussed as well as the critical phenomena arising if the external field strength comes into the range of order of the maximal self-field strength of the particle. It is found that the commonly accepted Lorentz-Dirac-Rohrlich theory is only a good approximation for small values of the external field strength.

I. What is the Problem?

There are numerous treatises in the literature, which try to overcome the infinities of the classical point-like electron by resorting to a non-linear generalization of Maxwell's electrodynamics *. The leading idea standing behind these attempts is the conjecture that Maxwell's linear theory is valid only in some distance of the center of the particle, where the field strength is sufficiently small. But in the immediate neighbourhood of the singularity of the linear theory the field strength should become so strong that the nonlinearities are dominating and are forcing down the field strength to a finite value in the center of the electron.

As an example consider the so-called "Rosen-particle" in scalar classical field theory ². The field equation in the static, spherical symmetric case is

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = -3g\Phi^5, \quad (\text{I},1)$$

which has as a particular, "particle-like" solution

$$\Phi = Z \cdot (Z^4 g + r^2)^{-1/2}. \quad (\text{I},2)$$

Clearly, the quantity

$$r_0^2 = Z^4 g \quad (\text{I},3)$$

is a characteristic linear dimension of the particle described by (I,2), and in the limit $r_0 \rightarrow 0$ one obtains Poisson's equation for the point-electron

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = -4\pi Z \delta(r). \quad (\text{I},4)$$

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* The most prominent attempt in this respect is that of Born and Infeld ¹.

Figures 1 and 2 show a plot of the potential $\Phi(r)$ and the field strength $E_r = -\partial\Phi/\partial r$.

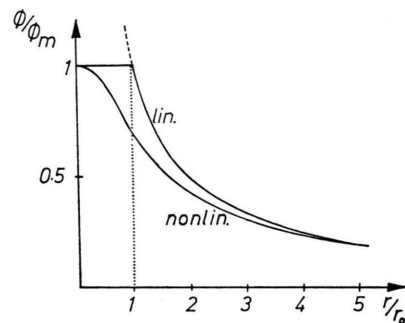


Fig. 1.

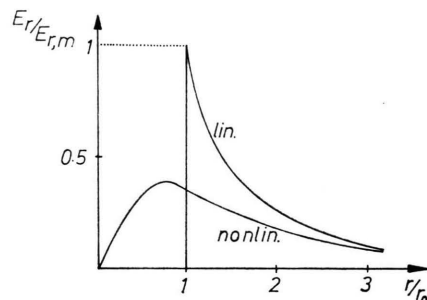


Fig. 2.

Now it is a well-known fact that non-linear theories are extremely difficult to handle with, not only as far as quantization is concerned but also with respect to the classical interaction with other particles or "external" fields. In order to get an impression what sort of phenomena arise in the classical theory as a lowest-order approximation to the nonlinearity effect, we have constructed a covariant



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cut-off procedure³, which consists in continuing the linear-theory potential A^0 of Maxwell's theory up to a sphere of radius r_0 around the singularity and giving the maximal value Z/r_0 of the nonlinear theory to the potential A^0 within this sphere (Fig. 1 and 2). Clearly, we identify the characteristic length r_0 with the classical electron radius (called Δs hereafter)

$$r_0 = Z^2/2 m_0 c^2, \quad (I,5)$$

which can be constructed from the experimental data of mass m_0 and charge Z for purely dimensional reasons.

One may suppose now, that there is some critical phenomenon in the behaviour of the finite-size electron, if it is subject to an external field which equals roughly the cut-off field strength $E_m = Z/r_0^2$. To study such an effect, simple enough to be treated exactly, is the object of the present paper.

Now, the theory, which is frequently used in the literature to deal with the radiating electron in a given external field, is the Lorentz-Dirac theory⁴ represented at full length in Rohrlich's book⁵. The basic equation in this theory is the Lorentz-Dirac equation

$$m_0 c^2 \ddot{u}^k = Z F_e^{rk} u_r + \frac{2}{3} Z^2 (\ddot{u}^k + (\dot{u} \dot{u}) u^k), \quad (I,6)$$

which is exactly solvable in the case of hyperbolic motion (constant, homogeneous, electric field E with no magnetic field present). Under these conditions, one gets from (I,6) for the invariant acceleration $k := \sqrt{-\dot{u} \dot{u}}$ in terms of the external field E a linear law

$$k = Z E / m_0 c^2, \quad (I,7)$$

which obviously does not exhibit any singular behaviour and therefore seems to be somewhat unrealistic for extremal high field-strengths E . Contrary to the result (I,7) we find in the present finite-size theory a nonlinear connection between k and E and a critical value of the invariant acceleration, which can not be surpassed. The corresponding critical field strength $E_c = \frac{1}{2} E_m$ is only available, if one brings two finite-size electrons in contact with each other. So we see that the finite-size theory brings out clearly its limit of validity; this limit arising wherever the interaction energy comes into the range of order of the rest energy of the particles involved. In this case the *cut-off* theory has to surrender to the *fully nonlinear* field theory, which is the more adequate theory for describing collision, annihila-

tion and creation processes by means of its "many-particle solutions"⁶.

II. Equation of Motion for the Radiating Electron

In a previous paper³ the following equation of motion for the finite-size electron has been deduced:

$$dP_b^\mu/d\tau + dP_{\text{rad}}^\mu/d\tau = K^\mu. \quad (II,1)$$

The bound part P_b^μ of the electromagnetic four-momentum is

$$P_b^\mu(z) = \frac{1}{c} \frac{Z^2}{2Q} \left\{ \frac{4}{3} \hat{u}^\mu (u \hat{u}) - \frac{1}{3} u^\mu \right\} \quad (II,2)$$

with

$$Q := (z - \hat{z}) \cdot u \quad (II,3)$$

being the acceleration-dependent electron radius, and

$$\hat{z} := z_{(s-\Delta s)}; \hat{u} := u_{(s-\Delta s)} \text{ etc.}$$

From (II,2) we see that the four-momentum $P_b^\mu(z)$ is a function of the world points z and \hat{z} by virtue of the finite size of the electron and that the notorious Schott-term emerges after a power-series expansion of P_b^μ in terms of the length parameter Δs , which we have put equal to the classical electron radius:

$$P_b^\mu \cong \frac{1}{c} \frac{Z^2}{2\Delta s} \cdot u^\mu - \frac{2}{3} \frac{Z^2}{c} \dot{u}^\mu. \quad (II,4)$$

Here is

$$m_0 c^2 = Z^2/2 \Delta s \quad (II,5)$$

the usual electron rest mass. Clearly, (II,2) says that there is no invariant constant rest mass in this usual sense but that the rest energy of the electron depends on the state of motion a (proper) time interval $\Delta s/c$ before the moment of the electron being at rest. So the possibility is indicated here that the part of P_b^μ which surpasses the usual expression $m_0 c u^\mu$ for P^μ possibly compensates in its temporal rate of change the energy-momentum \dot{P}_{rad}^μ radiated away from the particle.

Next we consider the radiation-recoil term in (II,1). In³ one has found

$$\dot{P}_{\text{rad}}^\mu(z) = - \frac{1}{c} \frac{2}{3} Z^2 (\hat{u} \hat{u}) \hat{u}^\mu. \quad (II,6)$$

It is the usual expression given in the literature in this context, but now shifted backwards the world line by the amount Δs according to the finite size of the electron.

Finally, one has for the electromagnetic force K^μ

$$K^\mu = \int (T_{\text{int}}^{\mu\nu} + T_e^{\mu\nu}) \cdot d^2 f_\nu \quad (\text{II},7)$$

with

$$d^2 f^\nu = (R_{\text{min}}^2 / (\hat{n} u)) \cdot d\hat{\Omega} \cdot \{ -u^\nu + [1 - \hat{u} \cdot (\xi - z)] \hat{n}^\nu \}. \quad (\text{II},8)$$

The integration in (II,7) with the surface element (II,8) ($\{\xi^\lambda\}$ is some point on the surface) has to be performed over the intersection of the future light cone (vertex in z) with the orthogonal hyperplane $\sigma_{\perp(z)}$ to the world line in the point z . The retarded distance R_{min} of this intersection from the world line is

$$R_{\text{min}} = \varrho / (\hat{n} u). \quad (\text{II},9)$$

As for the integrand in (II,7), one has

$$-4\pi T_{\text{int}}^{\mu\nu} = F_p^{\mu\lambda} F_{e\lambda}^\nu + F_e^{\mu\lambda} F_{p\lambda}^\nu - \frac{1}{2} g^{\mu\nu} (F_{p\alpha\beta} F_e^{\alpha\beta}) \quad (\text{II},10)$$

and

$$-4\pi T_e^{\mu\nu} = F_e^{\mu\lambda} F_{e\lambda}^\nu - \frac{1}{4} g^{\mu\nu} (F_e^{\alpha\beta} F_{e\alpha\beta}). \quad (\text{II},11)$$

Here is $F_e^{\mu\lambda}$ the external, electromagnetic force field, in which the electron is moving, and the self-field $F_p^{\mu\lambda}$ of the electron splits up into two parts

$$F_p^{\mu\lambda} = F_b^{\mu\lambda} + F_{\text{rad}}^{\mu\lambda} \quad (\text{II},12)$$

with

$$F_b^{\mu\lambda} = (2Z/R^2) \hat{u}^{[\mu} v^{\lambda]}, \quad (\text{II},13)$$

$$F_{\text{rad}}^{\mu\lambda} = (2Z/R) (\hat{u}^{[\mu} \hat{n}^{\lambda]} - (\hat{v} \cdot \hat{u}) \cdot \hat{u}^{[\mu} \hat{v}^{\lambda]}). \quad (\text{II},14)$$

In the previous work³ the connection between bound (resp. radiative) fields and bound (resp. radiative) four-momentum has been shown, and in the present context the splitting (II,12) has a surprising consequence for the finite-size electron, discussed in the next section.

In what follows it will be shown that hyperbolic motion is an exact solution of the equation of motion (II,1) in the case of a constant, uniform, external, electric field. To this purpose, we will provide ourselves with the exact expressions of P_b^μ , P_{rad}^μ , and K^μ for hyperbolic motion in the next section.

III. Special Forms of Forces in the Case of Hyperbolic Motion

1. Self-force

Since we do assume that hyperbolic motion is produced, even in the finite-size model, by a constant homogeneous electric field (no magnetic field pre-

sent) parallel to the direction of motion, we put

$$\{u^\lambda\} = \{\text{Cosh}(ks); 0, 0, \text{Sinh}(ks)\} \quad (\text{III},1) \\ (k = \text{const})$$

with the invariant acceleration k to be determined, and then find, that the following relations hold [$S := \text{Sinh}(kAs)$, $\mathcal{C} := \text{Cosh}(kAs)$]

$$\varrho = k^{-1} \cdot S, \quad (\text{III},2a)$$

$$(u \hat{u}) = \mathcal{C}, \quad (\text{III},2b)$$

$$(\hat{u} \hat{u}) = (\hat{u} \hat{u}) = -k^2, \quad (\text{III},2c)$$

$$(\hat{u} u) = -k \cdot S. \quad (\text{III},2d)$$

Therefore, all quantities on the left are constants of the motion. This simplifies the calculations extremely. Furthermore

$$\hat{u}^\lambda = \mathcal{C} \cdot u^\lambda - (k^{-1} S) \cdot \hat{u}^\lambda, \quad (\text{III},3a)$$

$$\hat{u}^\lambda = \mathcal{C} \cdot \hat{u}^\lambda - (k S) \cdot u^\lambda, \quad (\text{III},3b)$$

$$\hat{u}^\lambda = k^2 \cdot u^\lambda. \quad (\text{III},3c)$$

Now insert from here into (II,2) and (II,6) and find

$$\frac{dP_b^\mu}{d\tau} = \frac{1}{6} \frac{Z^2}{\varrho} \cdot \{ (4\mathcal{C}^2 - 1) \hat{u}^\mu - 4kS\mathcal{C} \cdot u^\mu \}, \quad (\text{III},4a)$$

$$\frac{dP_{\text{rad}}^\mu}{d\tau} = \frac{2}{3} Z^2 k^2 \cdot \{ \mathcal{C} \cdot u^\mu - k^{-1} S \cdot \hat{u}^\mu \}. \quad (\text{III},4b)$$

So we see very clearly, that the temporal rate of change of the energy of the bound field ($u_\mu \cdot dP_b^\mu/d\tau$) cancels the invariant radiation rate ($u_\mu \cdot dP_{\text{rad}}^\mu/d\tau$) and hence

$$\frac{dP_b^\mu}{d\tau} + \frac{dP_{\text{rad}}^\mu}{d\tau} = \frac{Z^2}{2\varrho} \cdot \hat{u}^\mu. \quad (\text{III},5)$$

This astonishing result says, that the nonlocal term of inertia combines in such a way with the radiation recoil, that the usual local term of inertia arises, if one identifies $Z^2/2\varrho$ with the rest energy $m_0 c^2$ of a nonradiating reference particle in the same force field. Since the two terms, which have cancelled away in (III,4a, b) \rightarrow (III,5), do not depend on S but on C , this statement remains valid even in the point limit $As \rightarrow 0$, where now the Schott-term would arise and would cancel the local radiation-recoil term. Clearly, the energy-momentum loss of the bound field has to be restored in those sections of the motion, where the motion is not hyperbolic, in order that the electron assumes its old rest mass before and after the acceleration process.

Finally let us point out, that the invariant radiation rate $(u_\mu \cdot dP_{\text{rad}}^\mu/d\tau)$ differs from that of the point-like electron by the factor \mathcal{C} , which is clearly due to the finite size of the electron, but it does not seem to be accessible to measurement for experimentally obtainable accelerations k .

2. External force

Defining in (II,7)

$$K_e^\mu := \int T_e^{\mu\nu} d^2 f_\nu \quad (\text{III,6})$$

our next task is to calculate exactly this purely external force. To do this, one can proceed in two ways:

a) direct calculation

As we deal with constant, homogeneous, external fields in the case of hyperbolic motion, the purely external force is, according to (III,6)

$$K_e^\mu = T_e^{\mu\nu} \int d^2 f_\nu. \quad (\text{III,7})$$

The integration in (III,7) with (II,8) is easily done, if we observe

$$\xi - z = -(z - \hat{z}) + R_{\min} \cdot \hat{n}, \quad (\text{III,8})$$

$$\dot{Q} = 1 - (u \hat{u}) + \dot{u} \cdot (z - \hat{z}) \quad (\text{III,9})$$

$$0 = \int T_{e,\nu}^{\mu\nu} d^4 V = \oint T_{e,\nu}^{\mu\nu} d^3 \sigma^\nu = \int_{(f)} T_{e,\nu}^{\mu\nu} d^3 f^\nu + \int_{\bar{\sigma}_{\perp(2)}} T_{e,\nu}^{\mu\nu} u_{(2)}^\nu d^3 \sigma_{\perp(2)} - \int_{\bar{\sigma}_{\perp(1)}} T_{e,\nu}^{\mu\nu} u_{(1)}^\nu d^3 \sigma_{\perp(1)}, \quad (\text{III,15})$$

where the three-volumes $\bar{\sigma}_{\perp(\dots)}$ are those which are cut out by the light cones $l_{(1)}$, $l_{(2)}$ from the orthogonal hyperplanes $\sigma_{\perp(1)}$, $\sigma_{\perp(2)}$. (III,15) reads in differential form ($d^3 f^\nu = d\hat{s} d^2 f^\nu$)

$$\int_{l(\hat{z}) \wedge \sigma_{\perp(z)}} T_{e,\nu}^{\mu\nu} d^2 f^\nu = \frac{d}{ds} \int_{\bar{\sigma}_{\perp(z)}} T_{e,\nu}^{\mu\nu} u_{(z)}^\nu d^3 \sigma_{\perp(z)} = T_{e,\nu}^{\mu\nu} \frac{d}{ds} \left\{ \frac{4\pi}{3} Q^3 u_{(z)}^\nu \right\}. \quad (\text{III,16})$$

and the integrals

$$\int d\hat{Q} [1/(\hat{n} u)^3] = 4\pi (u \hat{u}), \quad (\text{III,10})$$

$$\int d\hat{Q} [\hat{n}^\nu / (\hat{n} u)^3] = 4\pi u^\nu, \quad (\text{III,11})$$

$$\int d\hat{Q} [\hat{n}^\nu (\dot{u} \hat{n}) / (\hat{n} u)^4] = -\frac{4}{3} \pi \dot{u}^\nu. \quad (\text{III,12})$$

Then we find at once

$$\int d^2 f^\nu = 4\pi Q^2 \dot{Q} u^\nu + \frac{4\pi}{3} Q^3 \dot{u}^\nu = \frac{d}{ds} \left(\frac{4\pi}{3} Q^3 u^\nu \right), \quad (\text{III,13})$$

which is a total derivative and may be substituted in (III,7). The result thus obtained is readily verified

b) by means of Gauß' theorem

applied to the four-volume V_4 , which is bound by the three-dimensional "surface" (f) of the electron

and two orthogonal hyperplanes to the worldline (Figure 3).

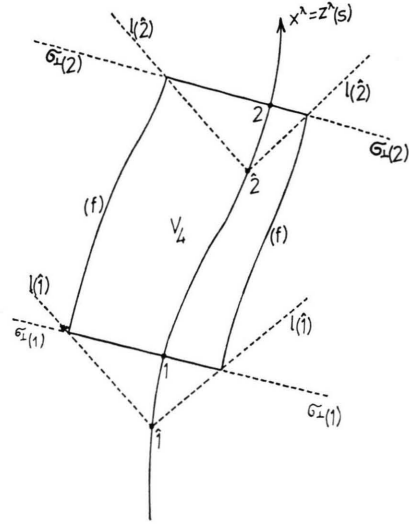


Fig. 3

With

$$T_{e,\nu}^{\mu\nu} \equiv 0 \quad (\text{III,14})$$

in the domain V_4 , Gauß' theorem now states

So we see, that the purely external force contribution depends linearly on the ordinary three-volume of the electron and vanishes, of course, in the point-particle limit. (Hint: Substitute $g^{\mu\nu}$ for $T_e^{\mu\nu}$ in (III,15) to obtain directly (III,13)).

Because the electron radius Q is a constant according to (III,2 a), it is finally found from (III,16) with (II,11) and (III,1)

$$K_e^\mu = \frac{4\pi}{3} Q^3 T_{e,\nu}^{\mu\nu} \dot{u}^\nu = \frac{1}{6} Q^3 E^2 \dot{u}^\mu, \quad (\text{III,17})$$

where we have put

$$-F_e^{30} = F_e^{03} =: E = \text{const}$$

and all other components of the external field-strength tensor vanishing.

3. Interaction force

Defining in (II,7)

$$K_{\text{int}}^\mu = \int T_{\text{int}}^{\mu\nu} d^2 f_\nu \quad (\text{III},18)$$

it is clear that one cannot proceed here as in the case of the external force K_e^μ , because $T_{\text{int}}^{\mu\nu}$ becomes singular on the world line due to the particle field $F_p^{\lambda\mu}$ contained in $T_{\text{int}}^{\mu\nu}$. Nevertheless, one has

$$T_{\text{int},\nu}^{\mu\nu} \equiv 0 \quad (\text{III},19)$$

off the world line, which is either seen from

$$T_{\text{tot},\nu}^{\mu\nu} \equiv 0; \quad T_{p,\nu}^{\mu\nu} \equiv 0; \quad T_{e,\nu}^{\mu\nu} \equiv 0 \quad (\text{III},20)$$

with

$$T_{\text{tot}}^{\mu\nu} = T_p^{\mu\nu} + T_{\text{int}}^{\mu\nu} + T_e^{\mu\nu} \quad (\text{III},21)$$

or from writing down explicitly $T_{\text{int}}^{\mu\nu}$ in terms of the external and particle potentials ($F^{\mu\nu} \equiv A^{\mu/\nu} - A^{\nu/\mu}$).

Therefore we can apply Gauß' integral theorem to the domain dV_4 , which is bound by the orthogonal hypersurfaces $\sigma_{\perp(1)}$, $\sigma_{\perp(2)}$, and the two tube surfaces (f') and (f'') arising from giving to the length parameter Δs two arbitrary values $\Delta s'$ and $\Delta s''$ between zero and its real value Δs (= classical electron radius) (Figure 4). Assuming the thickness of dV_4 infinitesimally small, we can write

$$\Delta s'' - \Delta s' = d(\Delta s') \quad (\text{III},22)$$

Gauß' theorem now yields

$$0 = \int_{dV_4} T_{\text{int},\nu}^{\mu\nu} d^4 x = \oint_{d\sigma} T_{\text{int}}^{\mu\nu} d^3 \sigma_\nu = - \int_{(f'')} T_{\text{int}}^{\mu\nu} d^3 f_\nu \quad (\text{III},23)$$

$$+ \int_{(f')} T_{\text{int}}^{\mu\nu} d^3 f_\nu + \int_{d\sigma_{\perp(2)}} T_{\text{int}}^{\mu\nu} u_{\nu(2)} d^3 \sigma_{\perp(2)} - \int_{d\sigma_{\perp(1)}} T_{\text{int}}^{\mu\nu} u_{\nu(1)} d^3 \sigma_{\perp(1)}.$$

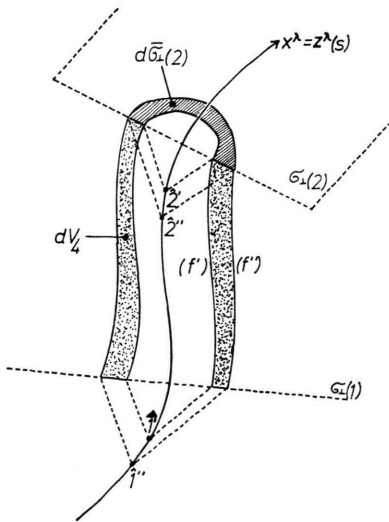


Fig. 4

If we now perform the limit $2 \rightarrow 1$ on the world line with $\Delta s'$ and $\Delta s''$ fixed, we find from (III,23) with the help of (III,18):

$$K_{\text{int}(\Delta s' + d(\Delta s'))}^\mu - K_{\text{int}(\Delta s')}^\mu \simeq \frac{dK_{\text{int}(\Delta s')}^\mu}{d(\Delta s')} d(\Delta s')$$

$$= \frac{d}{ds} \int T_{\text{int}}^{\mu\nu} u_{\nu(2)} d^3 \sigma_{\perp(2)} \quad (\text{III},24)$$

The integral on the right-hand side of this equation requires – by virtue of (II,10) – some integrals of the particle field running over the three-volume $d\bar{\sigma}_{\perp(2)}$ on $\sigma_{\perp(2)}$ (s. Figure 4). All these integrals are of the form

$$\int_{d\bar{\sigma}_{\perp(2)}} F_p^{\mu\lambda} d^3 \sigma_{\perp(2)} \quad (= \langle F_p^{\mu\lambda} \rangle) \quad (\text{III},25)$$

The integration (III,25) requires some work but is exactly feasible (Appendix):

$$\langle F_p^{\mu\lambda} \rangle = 8\pi Z I_{(\Delta s')} \cdot u^{[\mu} u^{\lambda]} \cdot d(\Delta s') \quad (\text{III},26)$$

with

$$I_{(\Delta s')} := \Delta s' \cdot \mathcal{C}' / S'^2 - 1/k \cdot S' \quad (\text{III},27)$$

Now we can substitute for $T_{\text{int}}^{\mu\nu}$ in (III,24) the expression (II,10) and can use for the arising integrals like (III,25) their values from (III,26) and then find

$$\frac{dK_{\text{int}(\Delta s')}^\mu}{d(\Delta s')} = \frac{d}{ds} \{ Z I_{(\Delta s')} \cdot F_e^{\mu\lambda} u_\lambda \} = Z k^2 I_{(\Delta s')} F_e^{\mu\lambda} u_\lambda \quad (\text{III},28)$$

Integrating (III,28) with respect to $\Delta s'$, one obtains

$$K_{\text{int}(\Delta s)}^\mu = K_{\text{int}(\Delta s=0)}^\mu - Z k^2 \tilde{I}_{(\Delta s)} F_e^{\mu\lambda} u_\lambda \quad (\text{III},29)$$

with

$$\tilde{I}_{(\Delta s)} := \int_0^{\Delta s} I_{(\Delta s')} d(\Delta s') = - \left(\frac{d}{dk} + \frac{1}{k} \right) \int_0^{\Delta s} \frac{d(\Delta s')}{\text{Sinh}(k \Delta s')}$$

$$= \frac{1}{k^2} \{ 1 - (k \Delta s) S^{-1} \}, \quad (\text{III},30)$$

where Δs is again the classical electron radius, according to our model assumptions.

Of course, one has to identify in (III,29)

$$K_{\text{int}(\Delta s=0)}^\mu = Z F_e^{\mu\lambda} u_\lambda, \quad (\text{III},31)$$

because if the tube (f) contracts up to the world line, the force (III,18) would degenerate to the usual Lorentz force (III,31) as is a well-known fact in literature (e.g. ³). (III,28) up to (III,31) finally yield now for the interaction force

$$K_{\text{int}}^\mu = Z (1 - k^2 \tilde{I}_{(\Delta s)}) F_e^{\mu\lambda} u_\lambda = Z S^{-1} (k \Delta s) E k^{-1} \dot{u}^\mu, \quad (\text{III},32)$$

where (III,1) has been used again.

IV. Hyperbolic Motion

With the results (III,5), (III,17), and (III,32) the equation of motion (II,1) can now be written as

$$((Z^2/2 \varrho) - \frac{1}{6} \varrho^3 E^2 - Z \Delta s S^{-1} E) \ddot{u}^\mu = 0. \quad (\text{IV},1)$$

Since this equation is valid in every point of the world line, we must have the expression in brackets vanishing, i.e.

$$E_{(k)} = 3 Z k^3 \Delta s S^{-4} \cdot \left\{ \sqrt{1 + \frac{1}{3} \cdot \frac{S^4}{(k \Delta s)^2}} - 1 \right\}. \quad (\text{IV},2)$$

($S = \text{Sinh}(k \Delta s)$)

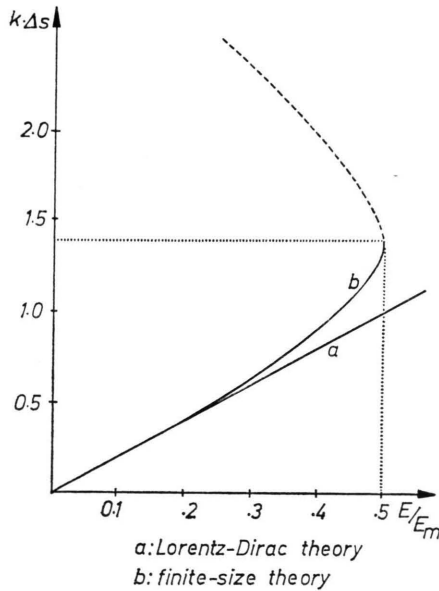


Fig. 5

Figure 5 shows a plot of this function. From this the following facts are readily established

1. The Lorentz-Dirac theory provides the tangent to the finite-size curve for $E \rightarrow 0$.
2. The deviation of the finite-size theory is always in the sense of greater values of k for a given E , i.e. the charged particle moves even faster than the neutral particle in the same force field, despite of the existence of the radiation recoil!
3. There exists a critical external electric field strength E_c

$$E_c \approx 0.5003 E_m = 0.5003 (Z/(\Delta s)^2), \quad (\text{IV},3)$$

beyond which our cut-off theory of the radiating finite-size electron becomes meaningless.

Let us inspect these results a little bit more thoroughly. First the third point: As can be seen from (IV,3), E_c is that electric field-strength, which is exerted on a point-like test charge in the distance $\tilde{\Delta s}$

$$\tilde{\Delta s} \approx \sqrt{2} \cdot \Delta s \approx 1.4 \cdot \Delta s \quad (\text{IV},4)$$

from the center of the finite-size electron. Now, with (III,2 a) one finds that the radius of the electron in its own rest frame is in the case of the maximal acceleration k_c (due to E_c)

$$\varrho_c = \text{Sinh}(k_c \Delta s) / k_c \approx 1.4 \cdot \Delta s. \quad (\text{IV},5)$$

Comparing (IV,4) and (IV,5) we see that our theory loses its validity if the particles come into contact with each other and this fact seems to be very plausible, because then creation and annihilation processes arise, the description of which falls into the domain of the fully nonlinear theory. So we see that our cut-off theory bears its own limit in itself (contrary to the Lorentz-Dirac theory).

The maximal invariant acceleration is

$$k_c \approx 1.4 \cdot (\Delta s)^{-1}$$

and therefore the maximal ordinary acceleration

$$\left| \frac{d\mathbf{v}}{dt} \right|_e = c^2 k_c \approx 9 \cdot 10^{33} \text{ [cm/sec}^2\text{]}$$

which is, of course, by no means available in linear accelerator technology.

Now let us examine the second point: Where does the energy come from to accelerate the radiating electron more rapidly than the nonradiating reference particle? From the left-hand side of (IV,1) one learns that the curvature of the finite-size line in Fig. 5 stems solely from the presence of the E^2 -term. This term is produced by the purely external force (III,17) and it is an immediate consequence of the finite size of the electron as shown by the argumentation following (III,16). If we would neglect this term in (IV,1), we would at once find the linear law (I,7) with (II,5) instead of (V,2). So we see, that the astonishingly simplified self-force term (III,5) together with the similarly simple interaction force (III,32) would alone not yield anything anew in the finite-size model compared to the Lorentz-Dirac theory. According to (III,6,16) one can interpret the purely external force as an additional energy-momentum content of the finite-size particle due to its presence in an external field.

The second term on the right of (IV,7), compensating the emission of electromagnetic four-momentum as shown by the argumentation following (III,5), has assumed (apart from the irrelevant \mathcal{C} being due to the finite-size of the electron) exactly the form of the Schott-term of the point-particle limit. Therefore the argumentation of the previous paper³ holds with respect to this point, i.e. the energy content of the bound field decreases in favour of the radiated energy. This is so, because the bound self-field, which contributes to the self-energy at a given moment, was emitted by the singularity at an earlier time, when the velocity and therefore also the energy content of the field emitted at that instant was smaller.

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Appendix

Calculation of $\langle F_p^{\mu\lambda} \rangle$

It is most convenient to express the three-cell $d^3\sigma_{\perp(z)}$ from (III,25) by its orthogonal projection $d^3\hat{\sigma}_{\perp}'$ on the orthogonal hyperplane $\sigma_{\perp}(\hat{z})$ to the world line in the point $\hat{z}': = z_{(s-\Delta s')}$

$$d^3\sigma_{\perp(z)} = \frac{d^3\hat{\sigma}_{\perp}'}{(u \cdot \hat{u}')} ; \quad (\hat{u}': = u_{(s-\Delta s')}) \quad (A,1)$$

and then integrate in $\sigma_{\perp}(\hat{z})$ over the projection $d\hat{\sigma}_{\perp}(\hat{z})$ of $d\hat{\sigma}_{\perp(z)}$

$$\langle F_p^{\mu\nu} \rangle = \frac{1}{(u \cdot \hat{u}')} \cdot \int F_p^{\mu\nu} d^3\hat{\sigma}_{\perp}'. \quad (A,2)$$

In order to find the three-cell $d^3\hat{\sigma}_{\perp}'$ in terms of the "polar coordinates" $\{R', \hat{\Theta}', \hat{\Phi}'\}$ introduced in the hyperplane $\sigma_{\perp}(\hat{z})$, we observe that the projection of the three-space $d\hat{\sigma}_{\perp(z)}$ (= space between two spheres in $\sigma_{\perp(z)}$ with radii ϱ and $\varrho + d\varrho$ respectively) on to $\sigma_{\perp}(\hat{z})$ is the space between two ellipsoids represented by

$$R' = \frac{\varrho(\Delta s')}{(u \cdot \hat{n}')} , \quad R'' = \frac{\varrho(\Delta s' + d(\Delta s'))}{(u \cdot \hat{n}')} \\ \cong \left(\varrho(\Delta s') + \frac{d\varrho}{d(\Delta s')} \cdot d(\Delta s') \right) / (\hat{n}' u) . \quad (A,3)$$

Because of

$$\frac{d\varrho(\Delta s')}{d(\Delta s')} = \frac{d}{d(\Delta s')} [u_{(s)} \cdot (z_{(s)} - z_{(s-\Delta s')})] = (u \cdot \hat{u}') \quad (A,4)$$

we have

$$dR' = R'' - R' = (u \cdot \hat{u}') \cdot d(\Delta s') \cdot \frac{1}{(\hat{n}' u)} \quad (A,5)$$

and therefore

$$d^3\hat{\sigma}_{\perp}' = d\hat{\Omega}' R'^2 (u \cdot \hat{u}') d(\Delta s') 1/(\hat{n}' u) \quad (A,6)$$

and finally with (III,25), (A.2) and (A.6)

$$\langle F_p^{\mu\nu} \rangle = d(\Delta s') \cdot \int \frac{d\hat{\Omega}' R'^2}{d\hat{\sigma}_{\perp}(\hat{z}')} \frac{1}{(u \cdot \hat{n}')} F_p^{\mu\nu} \quad (A,7)$$

with

$$R' = \varrho(\Delta s') / (u \cdot \hat{n}') . \quad (A,8)$$

As a consequence of the Liénard-Wiechert fields (II,12–14) the integration (A.7) requires the following integrals

$$\int d\hat{\Omega}' \frac{\hat{v}'^2}{(u \cdot \hat{n}')} = 4\pi a \left\{ u^2 - \frac{\mathcal{C}'}{k \cdot S'} \dot{u}^2 \right\}, \quad (A,9)$$

$$\int d\hat{\Omega}' \frac{\hat{v}'^2}{(u \cdot \hat{n}')^2} = 4\pi b \left\{ u^2 - \frac{\mathcal{C}'}{k \cdot S'} \dot{u}^2 \right\}, \quad (A,10)$$

$$\int d\hat{\Omega}' \frac{(\hat{v}' \cdot \hat{u}') \hat{v}'^2}{(u \cdot \hat{n}')^2} = 4\pi k f \left\{ u^2 - \frac{\mathcal{C}'}{k \cdot S'} \dot{u}^2 \right\}, \quad (A,11)$$

$$\int d\hat{\Omega}' \frac{1}{(\hat{n}' u)^2} = 4\pi \quad (A,12)$$

where

$$a = 1 - k \Delta s' \cdot \frac{\mathcal{C}'}{S'} ; \quad b = \frac{k \Delta s'}{S'} - \mathcal{C}' ; \quad (A,13)$$

$$f = \frac{1 + \mathcal{C}'^2}{S'} - 2(k \Delta s') \cdot \frac{\mathcal{C}'}{S'^2}$$

and

$$S' := \text{Sinh}(k \Delta s') ; \quad \mathcal{C}' := \text{Cosh}(k \Delta s') .$$

We are exhibiting the technique of calculation for the first integral (A.9); the others are done in the same way. Choosing for the four-velocity

$$\{u^i\} = \{\text{Cosh } k(s - \hat{s}'); 0, 0, \text{Sinh } k(s - \hat{s}')\} , \quad (A,14)$$

the electron is obviously at rest in the point $\hat{z}' = z_{(s')}$. A time interval $(\Delta s')$ later, the four-velocity has components

$$\{\mathcal{C}'; 0, 0, S'\} \quad (A,15)$$

and hence the only non-vanishing component of (A.9) becomes

$$\int d\hat{\Omega}' \frac{\hat{v}'^3}{(u \cdot \hat{n}')} = \int d\hat{\Phi}' \int \sin \hat{\Theta}' d\hat{\Theta}' \frac{\cos \hat{\Theta}'}{\{\mathcal{C}' - S' \cdot \cos \hat{\Theta}'\}} \\ = -4\pi \left(\frac{1}{S'} - (k \Delta s') \cdot \frac{\mathcal{C}'}{S'^2} \right) . \quad (A,16)$$

Therefore

$$\int d\hat{\Omega}' \frac{\hat{v}'^\lambda}{(u \cdot \hat{n}')^2} = -4\pi \left(\frac{1}{S'^2} - (k \Delta s') \cdot \frac{\mathcal{C}'}{S'^3} \right) \cdot \{u^\lambda - (u \hat{u}') \hat{u}'^\lambda\} \quad (\text{A},17)$$

and with the relations (III,2) and (III,3) the desired integral (A.9) follows.

Now from (II,13) and (II,14)

$$\begin{aligned} \frac{\langle F_p^{\mu\lambda} \rangle}{d(\Delta s')} &= 2Z \hat{u}'^{[\mu} \int d\hat{\Omega}' \frac{\hat{v}'^{\lambda]}}{(u \hat{n}')^2} \\ &+ 2Z \mathcal{Q}_{(\Delta s')} \hat{u}'^{[\mu} \hat{u}'^{\lambda]} \int d\Omega' \frac{1}{(u \hat{n}')^2} \\ &+ 2Z \mathcal{Q}_{(\Delta s')} \hat{u}'^{[\mu} \int d\hat{\Omega}' \frac{\hat{v}'^{\lambda]}}{(\hat{n}' u)^2} \\ &- 2Z \mathcal{Q}_{(\Delta s')} \hat{u}'^{[\mu} \int d\hat{\Omega}' \frac{(\hat{v}' \hat{u}') \hat{v}'^{\lambda]}}{(u \hat{n}')^2} \end{aligned} \quad (\text{A},18)$$

and from here with the integrals (A.9–12) and the relations (III,3 a, b)

$$\begin{aligned} \frac{\langle F_p^{\mu\lambda} \rangle}{d(\Delta s')} &= 8\pi Z \left\{ \frac{a}{k S'} + \mathcal{Q}_{(\Delta s')} - \mathcal{Q}_{(\Delta s')} \frac{f}{S'} \right\} \hat{u}^{[\mu} u^{\lambda]} \\ &\equiv 8\pi Z I_{(\Delta s')} \hat{u}^{[\mu} u^{\lambda]}, \end{aligned} \quad (\text{A},19)$$

where we have abbreviated

$$I_{(\Delta s')} := \frac{a}{k S'} + \mathcal{Q}_{(\Delta s')} - \mathcal{Q}_{(\Delta s')} \frac{f}{S'} = \Delta s' \frac{\mathcal{C}'}{S'^2} - \frac{1}{k \cdot S'}.$$

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